Olympiad 1

1. (4 min.)

Suppose today is Tuesday. What day of the week will it be 100 days from now?

2. (5 min.)

I have four 3¢-stamps and three 5¢-stamps. Using one or more of these stamps, how many different amounts of postage can I make?

3. (5 min.)

Find the sum of the counting numbers from 1 to 25 inclusive. In other words, if $S = 1 + 2 + 3 + \cdots + 24 + 25$, find the value of S.

4. (6 min.)

In a stationery store, pencils have one price and pens have another price. Two pencils and three pens cost 78¢. But three pencils and two pens cost 72¢. How much does one pencil cost?

5. (5 min.)

A work crew of 3 people requires 3 weeks and 2 days to do a certain job. How long would it take a work crew of 4 people to do the same job if each person of both crews works at the same rate as each of the others? Note: each week contains six work days.

Olympiad 2

1. 4 min.	A girl bought a dog for $10, sold it for $15, bought it back for $20, and finally sold it for $25. Did the girl make or lose money, and how much did she make or lose?
2. 5 min.	I have 30 coins consisting of nickels and quarters. The total value of the coins is $4.10. How many of each kind do I have?
3. 5 min.	Rectangular cards, 2 inches by 3 inches, are cut from a rectangular sheet 2 feet by 3 feet. What is the greatest number of cards that can be cut from the sheet?
4. 5 min.	In three bowling games, Alice scores 139, 143, and 144. What score will Alice need in a fourth game in order to have an average score of 145 for all four games?
5. 6 min.	A book has 500 pages numbered 1, 2, 3, and so on. How many times does the digit 1 appear in the page numbers?

Olympiad 3

1\. (4 min.)

A set of marbles can be divided in equal shares among 2, 3, 4, 5, or 6 children with no marbles left over. What is the least number of marbles that the set could have?

2\. (5 min.)

A motorist made a 60-mile trip averaging 20 miles per hour. On the return trip, he averaged 30 miles per hour. What was the motorist's average speed for the entire trip?

3\. (4 min.)

The four-digit numeral 3AA1 is divisible by 9.What digit does A represent?

4\. (7 min.)

Express the following sum as a simple fraction in lowest terms.

$$\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\frac{1}{4\times 5}+\frac{1}{5\times 6}$$

5\. (5 min.)

If we count by 3s starting with 1, the following sequence is obtained: 1, 4, 7, 10, ... What is the 100th number in the sequence?

Olympiad 4

1. (5 min.)

100 pounds of chocolate is packaged into boxes each containing $1\frac{1}{4}$ pounds of chocolate. Each box is then sold for \$1.75. What is the total selling price for all of the boxes of chocolate?

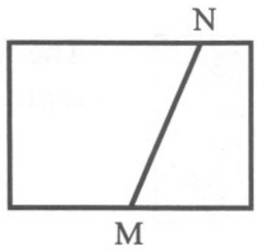
2. (5 min.)

In the multiplication problem at the right, A and B stand for different digits. Find A and B.

$$\begin{array}{r} A\ B \\ \times\ B\ A \\ \hline 1\ 1\ 4 \\ 3\ 0\ 4\ \ \\ \hline 3\ 1\ 5\ 4 \end{array}$$

3. (5 min.)

In the rectangle at the right, line segment MN separates the rectangle into 2 sections. What is the largest number of sections into which the rectangle can be separated when 4 line segments are drawn through the rectangle?



4. (6 min.)

If $\frac{1}{3} = \frac{1}{A} + \frac{1}{B}$ where A and B are different whole numbers, find the value of A and the value of B.

5. (5 min.)

P and Q represent numbers, and P * Q means $\frac{P+Q}{2}$. What is the value of 3 * (6 * 8)?

Olympiad 5

1. (4 min.)

The numbers 2, 4, 6, and 8 are a set of four consecutive even numbers. Suppose the sum of five consecutive even numbers is 320. What is the smallest of the five numbers?

2. (5 min.)

Amy can mow 600 square yards of grass in 1½ hours. At this rate, how many minutes would it take her to mow 600 square feet?

3. (6 min.)

Express the extended fraction at the right as a simple fraction in lowest terms.

$$\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2}}}}$$

4. (5 min.)

There are many numbers that divide 109 with a remainder of 4. List all two-digit numbers that have that property.

5. (6 min.)

A dealer packages marbles in two different box sizes. One size holds 5 marbles and the other size holds 12 marbles. If the dealer packaged 99 marbles and used more than 10 boxes, how many boxes of each size did he use?

Hints

Olympiad 1

1) What day will it be 7 days from now? 14 days from now? 77 days from now?
2) Make an organized list of the different amounts starting with the 3¢-stamps.
3) Rewrite the series in reverse order placing each term directly under the term of the given series. Examine each vertical pair of terms.
4) How much will 5 pens and 5 pencils cost?
5) How long would it take one person to do the entire job alone?

Olympiad 2

1) Act it out.
2) Try using half of the coins as nickels and the other half as quarters.
3) How many square inches are there in the rectangular sheet 2 feet by 3 feet?
4) If the average score for 4 games is 145, what is the total score for the 4 games?
5) If you counted from 1 on, how frequently would "1" appear in the units place? tens place? hundreds place?

Olympiad 3

1) Try a simpler problem with 2, 3, or 4 children.
2) Average speed is the total distance divided by the total time.
3) If the sum of the digits of a number is divisible by 9, the number is also divisible by 9.
4) $\frac{1}{9 \times 10} = \frac{1}{9} - \frac{1}{10}$.
5) Compare the terms of the sequence with multiples of 3, starting with 3.

Olympiad 4

1) Make 1.75 and 1¼ either both decimals or both mixed numbers.
2) $A \times AB = 114$. Try different values for A starting with 2.
3) Experiment with 2 lines and count the sections. Then try 3 lines, and then 4 lines.
4) Could A be less than 3?
5) Do (6*8) first.

Olympiad 5

1) What is the average of the five numbers?
2) How many times larger than 600 square feet is 600 square yards?
3) Work from the bottom up.
4) What is the largest number that the two-digit numbers can divide exactly?
5) Try packaging some marbles in the larger boxes and examine what is left over.

Solutions

Olympiad 1

1) Every 7 days from "today" will be Tuesday. Since 98 is a multiple of 7, the 98th day from today will be Tuesday. Then the 100th day from today will be Thursday.

2) Method 1
List the amounts in an organized manner.

	Amounts				Number
Amounts from 3¢-stamps:	3,	6,	9,	12	4
Amounts from 5¢-stamps:	5,	10,	15		3
Amounts from combining					
3¢-stamps and 5¢-stamps:	3+5,	3+10,	3+15		3
	6+5,	6+10,	6+15		3
	9+5,	9+10,	9+15		3
	12+5,	12+10,	12+15		3
				Total	19

Method 2
The number of choices we have in using the 3¢-stamps is 5; we can use either 0, 1, 2, 3, or 4 of the 3¢-stamps. Similarly, we have 4 choices with respect to the 5¢-stamps; we can use either 0, 1, 2, or 3 of the 5¢-stamps. Each of the 5 choices for the 3¢-stamps can be combined with one of the four choices we have for the 5¢-stamps. This gives a total of 20 combinations. However, this total includes the combination of 0 3¢-stamps and 0 5¢-stamps. Since 1 or more of the stamps must be used, we exclude the combination of none of each. Therefore 19 different amounts of postage can be made.

3) Method 1

Given	(1)	$S = 1 + 2 + 3 + \cdots + 23 + 24 + 25$
Reverse order of right side of (1)	(2)	$S = 25 + 24 + 23 + \cdots + 3 + 2 + 1$
Add (1) and (2)	(3)	$2S = 26 + 26 + 26 + \cdots + 26 + 26 + 26$
Simplify the right side of (3)	(4)	$2S = 26 \times 25$
Divide both sides of (4) by 2	(5)	$S = 13 \times 25$ or 325

The required sum is 325.

Method 2
Arrange the numbers in a square array as shown. Add the numbers in the left column (or bottom row). The sum of each of the other columns (or rows) can be easily determined by inspection. For example, each number in the second column is one more than its corresponding number in the first column. This is also true for other pairs of successive columns. The sum of the 5 columns (or rows) is 325.

21	22	23	24	25	115
16	17	18	19	20	90
11	12	13	14	15	65
6	7	8	9	10	40
1	2	3	4	5	15
55	60	65	70	75	325

Solutions

(Olympiad 1)

3) Method 3

The sum of the numbers in the third column (or in the third row) is 65. This is the average of the sums of the five columns (or rows). Multiply 65 by 5 to get the complete sum: $65\times5 = 325$.

Method 4

Examine the array of numbers shown in Method 2. Observe that the average of all numbers is 13, the number in the middle of the array. Then the sum must be $13\times25 = 325$.

4) Method 1

By combining both purchases we find that 5 pencils and 5 pens cost 150¢. Then 1 pencil and 1 pen cost 30¢, or 2 pencils and 2 pens cost 60¢. Since 3 pencils and 2 pens cost 72¢, 1 pencil costs 12¢.

Method 2

The difference in the prices of the two purchases is equivalent to the difference in the costs of a pen and a pencil. Therefore, a pen costs 6¢ more than a pencil, or, 3 pens cost 18¢ more than 3 pencils. Thus, the first purchase of 2 pencils and 3 pens is equivalent to the purchase of 2 pencils and 3 pencils plus 18¢, or 5 pencils plus 18¢. Since the cost of this purchase was 78¢, 5 pencils alone cost 60¢. Therefore 1 pencil had a cost of 12¢.

Method 3

Algebra: Let C represent the cost of one pencil and N the cost of one pen.

Given	(1)	2C	+	3N	=	78
Given	(2)	3C	+	2N	=	72
Multiply both sides of (2) by 3	(3)	9C	+	6N	=	216
Multiply both sides of (1) by 2	(4)	4C	+	6N	=	156
Subtract (4) from (3)	(5)	5C			=	60
Divide both sides of (5) by 5	(6)			C	=	12

Answer: A pencil costs 12¢

5) Each person of the work crew of three people worked 20 days. Thus the number of individual work days needed to do the job was 60. Then each member of the work crew of four people must work 15 days in order to provide a total of 60 individual work days.

Solutions

Olympiad 2

1) She paid out \$10 + \$20 = \$30. She received \$15 + \$25 = \$40. She made \$10.

2) Method 1

Make a table. Let N represent the number of nickels and Q the number of quarters. Start with 30 nickels and 0 quarters. Then in each successive line, decrease the number of nickels by 1 and increase the number of quarters by 1 thus keeping the total number of coins as 30. Notice that every time we decrease the number of nickels by 1 and increase the number of quarters by 1, we increase the preceding total value by 20¢. Observe that the number of increases of 20 in the total value column is the same as the number of quarters on the same line. To increase 150 in the total value column to 410, we need to increase 150 by 260 or 13 × 20. Thus there were 13 quarters and 17 nickels in the 30 coins.

N	Q	Total value in ¢
30	0	150
29	**1**	170 = 150 + **1** × 20
28	**2**	190 = 150 + **2** × 20
27	**3**	210 = 150 + **3** × 20
...	...	... ...
17	**13**	410 = 150 + **13** × 20

Method 2

Let us look at the same problem in a different setting. Suppose 30 spiders had a total of 410 legs. Suppose further that some of the 30 spiders (nickelpeds) each have 5 legs and the others (quarterpeds) each have 5 rear legs and 20 forelegs. Suppose the nickelpeds always have their legs on the ground and the quarterpeds have just their 5 rear legs on the ground. Then the 30 spiders have just 150 legs on the ground and 260 legs in the air. But each quarterped has 20 legs in the air. Then there must be 260/20 = 13 quarterpeds. (Now compare method 1 and method 2; notice the similarities.)

Method 3

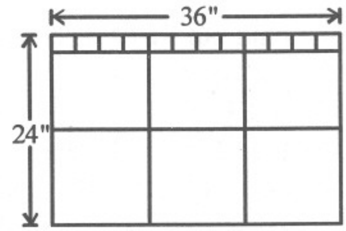
Algebra: Let N represent the number of nickels and Q the number of quarters.

Given	(1)	N	+	Q	=	30
Given	(2)	5N	+	25Q	=	410
Divide both sides of (2) by 5	(3)	N	+	5Q	=	82
Subtract (1) from (3)	(4)			4Q	=	52
Divide both sides of (4) by 4	(5)			Q	=	13
Substitute from (5) into (1)	(6)	N	+	13	=	30
Subtract 13 from both sides of (6)	(7)			N	=	17

Therefore there are 17 nickels and 13 quarters in the 30 coins.

Solutions

(Olympiad 2)



3) Method 1

Make a diagram of the sheet showing one 2" strip. Notice that there are twelve 2" by 3" pieces in the 2" strip. Since there are 12 strips of 2" in 24", there are $12\times12 = 144$ 2" by 3" pieces in the 2' by 3' rectangle.

Method 2

The area of one card is 6 square inches. The area of the sheet is 24×36 square inches. Divide 24×36 by 6 to produce 144.

4) Method 1

If the average of four scores is 145, then their sum is $4 \times 145 = 580$. The sum of the three given scores is $139 + 143 + 144 = 426$. The fourth score is equal to $580 - 426$ or 154.

Method 2

The first score is 6 less than the average; the second score is 2 less than the average; and the third score is 1 less than the average. Thus the sum of the three scores is 9 less than the sum of three average scores. Thus the fourth score needs to be 9 above the average score which is $145 + 9 = 154$.

Method 3

Algebra: Let S represent the fourth score.

Definition of average	(1)	$(139 + 143 + 144 + S) \div 4 = 145$
Multiply both sides of (1) by 4	(2)	$139 + 143 + 144 + S = 145 \times 4$
Simplify (2)	(3)	$426 + S = 580$
Subtract 426 from both sides of (3)	(4)	$S = 154$

Therefore the fourth score is 154.

5) Consider the frequency of appearance of the digit "1" in each of the places.

units place: the digit "1" appears once in every ten. Since 500 has 50 tens, the digit "1" will appear 50 times in the units place.

tens place: the digit "1" appears ten times in every hundred. Since 500 has 5 hundreds, the digit "1" will appear 50 times in the tens place.

hundreds place: the digit "1" will appear 100 times in the hundreds place (100, 101, 102, ... , 199)

The digit "1" will appear a total of 200 times in the page numbers.

Solutions

Olympiad 3

1) The least number of marbles that the set could have is the least common multiple of 2, 3, 4, 5, and 6. LCM(2,3,4,5,6) = 60. (See Appendix 6, section 3.)

2) The average speed for any trip is the total distance divided by the total time spent in traveling. The total distance was 120 miles and the total time was 5 hours. The average speed equals (120 miles)/(5 hours) or 24 miles/hour or 24 mph.

Comment: It is interesting to observe that the average speed in this problem does not depend on the distance. In other words, the average speed in this problem would be the same no matter what distance was traveled at the given rates.

3) If a number is divisible by 9, then the sum of its digits is divisible by 9. The digit sum is 3+A+A+1 = 4+2A. The digit sum cannot be 9, otherwise A = 2½. So 4+2A = 18 which produces A = 7. (See Appendix 4, section 4.)

4) Any unit fraction whose denominator is the product of two consecutive numbers can be expressed as a difference of unit fractions as shown at the right. The second equation is the general rule.

$$\frac{1}{99\times100}=\frac{1}{99}-\frac{1}{100}$$
$$\frac{1}{n(n+1)}=\frac{1}{n}-\frac{1}{n+1}$$

Each of the fractions in the given sum can be expressed as the difference of two unit fractions like so:

$$\left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\left(\frac{1}{4}-\frac{1}{5}\right)+\left(\frac{1}{5}-\frac{1}{6}\right)$$

Observe that when the addition is performed, all terms but the first and last drop out. Therefore the sum is 1 – 1/6 or 5/6.

5) Since the successive terms of the sequence increase by 3, relate the sequence to the multiples of 3 and display the table at the right. Compare each of the terms of the sequence with the multiple of 3 directly below it. Notice that each term of the sequence is 2 less than the corresponding multiple of 3 directly below it. Since the 100th multiple of 3 is 300, the corresponding term of the sequence is 300 – 2 = 298.

order of terms	1	2	3	4	. . .	100
terms of sequence	1	4	7	10	. . .	?
multiples of 3	3	6	9	12	. . .	(300)

Solutions

Olympiad 4

1) We need to know the number of boxes that were packaged in order to find the total selling price. The number of boxes is obtained by dividing 100 by 1¼: $100 \div 5/4 = 100 \times 4/5 = 80$. The selling price is $80 \times \$1.75 = \$140$.

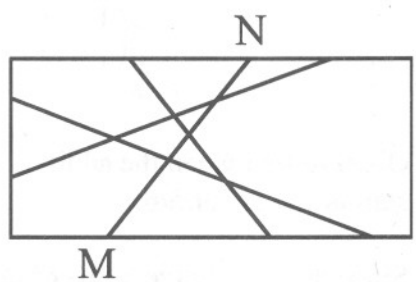
2) The first partial product 114 is equal to the product of AB and A. The second partial product 304 is equal to the product of AB and B. Then A must be less than B.

Method 1
Since the product of AB and A is 114, A is a divisor of 114. Therefore A may be 2, 3, or 6. Since AB×A = 114, A cannot be 2 because AB×A would then be less than 60. Similarly, A cannot be 6 since AB×A would then be greater than 360. Therefore A must be 3 and AB must be 114/3 or 38. Thus A = 3 and B = 8.

Method 2
From the first partial product, observe that B×A must end in 4. Since A is less than B, A = 2 and B = 7, or A = 3 and B = 8, or A = 4 and B = 6. But 27×2 = 54, 46×4 = 184, and 38×3 = 114. Only the third equation satisfies the given conditions. So A = 3 and B = 8.

Method 3
From the second partial product 304, we see that B×B ends in 4. Then B = 2 or 8. If B = 2, then A must be 1 because A is less than B. But 12×21 = 252. If B = 8 and AB×B = 304, then AB = 304/8 or 38 and 38×83 = 3154. Therefore A = 3 and B = 8.

3) Method 1
Make a diagram and draw 4 lines so that they intersect each other as shown. The number of different sections is 11.



Method 2
Make a table. The original rectangle without lines added is considered to be one section.

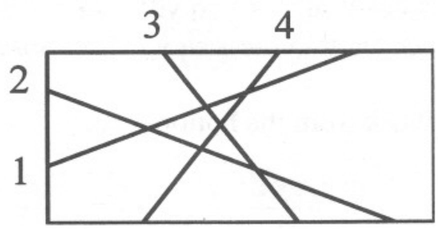
total number of lines added	0	1	2	3	4
total number of sections	1	2	4	7	?

Solutions

(Olympiad 4)

3) Method 2

Look for a pattern. Observe that the 1st added line results in increasing the preceding total of sections by 1, the 2nd added line increases the preceding total of sections by 2, the 3rd added line increases the preceding total of sections by 3. It seems that the 4th added line will increase the preceding total of sections by 4. and that there will be 7+4 or 11 sections. Examine the 4th line in the diagram at the right. When the 4th line intersects the first of the 3 interior lines, it creates a new section. This happens each time the 4th line crosses an interior line. When the 4th line finally ends at a point on the rectangle, it creates a 4th new section. Thus the 4th line creates a total of 4 new sections. The answer to the given problem is 11. (If a 5th line were added, it would increase the preceding total of sections by 5.)



4) A has to be larger than 3. Try A = 4: $\frac{1}{3}=\frac{1}{4}+\frac{1}{B}$ or $\frac{1}{B}=\frac{1}{3}-\frac{1}{4}=\frac{1}{12}$
Therefore A = 4 and B = 12.

The following rule applies to all unit fractions: $\frac{1}{N} = \frac{1}{N+1} + \frac{1}{N(N+1)}$.

5) According to the order of operations, first perform the operation indicated in the parentheses.

$$6 * 8 = \frac{6+8}{2} \text{ or } 7. \quad \text{Then} \quad 3 * (6 * 8) = 3 * 7 = \frac{3+7}{2} \text{ or } 5.$$

Olympiad 5

1) Method 1

The middle number of an odd number of consecutive numbers is always the average of the set. Then the average of the numbers is 320/5 or 64 which also is the third or middle number. Count back by twos. The required number is 60.

Method 2

Represent the middle number by n. Then the five consecutive even numbers are n–4, n–2, n, n+2, and n+4. The sum of the five numbers is 5n. Since 5n = 320, n = 64. Thus n–4, the first number, is 60.

2) Method 1

1 sq yd = 9 sq ft. Then 600 sq yd = 600×9 sq ft. Since 600 sq ft is 1/9 of 600×9 sq ft, the time needed to mow 600 sq ft is 1/9 of the time required to mow 600 sq yd.
herefore 1/9 of 1½ hrs is 1/9 × 3/2 = 1/6 hr or 10 min.

Solutions

(Olympiad 5)

Method 2
Since 9 sq ft = 1 sq yd, 1 sq ft = 1/9 sq yd. Then the time needed to mow 1 sq ft is 1/9 of the time needed to mow 1 sq yd. Therefore, 600 sq ft will require 1/9 of 1½ hr or 1/6 hr or 10 min.

3) Work from the bottom up.

$$\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2}}}} = \cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{5/2}}} = \cfrac{1}{2+\cfrac{1}{2+\frac{2}{5}}} = \cfrac{1}{2+\cfrac{1}{12/5}} = \cfrac{1}{2+\frac{5}{12}} = \frac{1}{29/12} = \frac{12}{29}$$

4) If 4 is subtracted from 109, the result is 105. Then each of the two-digit numbers that will divide 109 with a remainder of 4 will divide 105 exactly. Thus, the problem is equivalent to finding all two-digit divisors of 105. Since the prime factors of 105 are 3, 5, and 7, the divisors are 3×5, 3×7, and 5×7, or 15, 21, and 35.

5) Method 1
After some marbles are packaged in boxes for 12, the remaining marbles must be completely packaged in boxes for 5. The following table shows what happens when some marbles are packaged in boxes for 12. Only in two cases (marked with an asterisk) can all 99 marbles be completely packaged in 12-marble and 5-marble boxes. However, only the first of these two *cases will satisfy the condition that more than 10 boxes must be used.

Number of boxes for 12	Number of extra marbles	Number of boxes for 5	Total number
1	99 – 12 = 87	17; 2 m left over	--
*2	99 – 24 = 75	15	17
3	99 – 36 = 63	12; 3 m left over	--
. . .	. . .	. . .	. . .
*7	99 – 84 = 15	3	10

Therefore 2 of the large boxes and 15 of the small boxes were used.

Method 2
Let S represent the number of 5-marble boxes and L the number of 12-marble boxes.

Then $5S + 12L = 99$ or $S = \frac{99-12L}{5}$. In the second equation, 99 – 12L must be divisible by 5 if S is to be a whole number. This will happen only if L = 2 or 7. If L = 2, S = 15. If L = 7, S = 3. The number of boxes will be greater than 10 only when L = 2 and S = 15.